***Conclusion.*** *Modeling the behavior of pseudo-elastic-plastic material with large plastic deformations requires the use of nonlinear mathematical models that could more accurately describe and predict the behavior of such a body. The behavior of the locally loaded weakened strip of pseudo-elastic-plastic material under its non-stationary loading is modeled in the work. The authors used a nonlinear phenomenological model of the material to solve the above problem, which allows to describe a number of experimental data on different samples under different conditions. A comparison of the results obtained in geometrically linear and nonlinear formulations with large plastic deformations. Numerically compared intensity fields at symmetric and asymmetric loading It is established that at plastic deformations up to 6% (small deformations) the discrepancy of results in points of localization of deformation does not exceed 5%. With increasing values of plastic deformation (more than 7%, large deformations), the discrepancy of the results can increase significantly and in the vicinity of the possible creation of the neck to reach 20%.*

***Keywords:*** *phenomenological model, nonlinear model of material, materials with memory of forms, thermo-pseudo-plasticity, large plastic deformations.*

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## NUMERICAL-ANALYTICAL METHOD FOR THE PROBLEMS OF ENVIROMENTAL SAFETY

*In problems of modeling pollution of water and atmospheric resources of the Earth’s ecosystems, models of potential flow are often used. Recently, considerable attention has also been paid to protecting metropolis from atmospheric pollution with acoustic noise. Very often, without focusing on local flow features, the Laplace equation is used to describe the potential equation motion of a fluid. Acoustic problems are modeled based on the Helmholtz equation.*

*In the work presented below, the features of the numerical-analytical method for the Helmholtz and Laplace equations are considered. The sound potential is a rapidly oscillating function given at the boundary of the computational domain. In addition, the numerically-analytical method peculiarities are presented for the Laplace equation that uses the expansion of the boundary condition in a Fourier series in eigenvalues of the Sturm-Liouville problem. Despite the fact that the data presented in the work were obtained for canonical domains, the scheme of the method implies the possibility of using it for domains with an arbitrary curvilinear boundary. The numerical-analytical method proposed in the paper allows for low computational costs to solve numerically the problems for the Laplace equation and the acoustic equations. The results of the research conducted can be used as new information technology to address the environmental security challenges of water and air resources.*

***Keywords:*** *Computational modeling, Numerical method, Geo-environmental monitoring.*

### Introduction

One of the major problems of today is the problem of preventing pollution of the Earth's water and air resources. It is directly related to the geo-environmental monitoring of the planet. To prevent global man-made disasters, scientists are developing mathematical models that describe the processes of occurrence and spread of pollution in aquatic environments.

There is also the problem of protecting the environment from noise pollution by helicopters. Recently, considerable attention has been paid to the development of information technologies in the field of environmental safety. In particular, much attention is paid to the study of processes of pollution of water resources and the atmosphere, as well as to the acoustic pollution (noisiness) of megapolices. In civil aviation [1], in the operation of helicopters, very often unforeseen situations occur that result in local environmental disasters that lead to undesired environmental pollution.

It is known that in most situations, the main equation for modeling fluid motion is the Laplace equation [2], [3]. Modeling of aerodynamic noise, helicopter noise, is performed on the basis of the equation of propagation of small perturbations from a thin wing [4]. Numerical methods and algorithms for its solution are constantly being improved. Numerical circuits are being searched for, allowing smaller computer resources to achieve the desired result. In this paper, we consider a numerical method for numerical modeling these processes. The numerical-analytical method is an implicit analogue of the finite-difference method: in the finite-difference method, all derivatives are a priori expressed in terms of values in the calculation nodes, and in the numerical-analytical method they are expressed implicitly directly during the solution of a specific differential equation, a boundary-value problem. Thus, this method allows you to take into account the specifics of the differential operator the behavior of the function on the boundary.

In a number of problems of hydromechanics, one has to deal with the Laplace equation:

$$\Delta f = 0\,, \tag{1}$$

where $f \in C^2(A)$, $A \subset R^2$.

If the problem is solved in the canonical domain, for example, in a rectangular 2-dimensional domain, then the desired solution based on the Fourier method is represented as a trigonometric series in eigenvalues functions. A similar situation holds for the Helmholtz equation. The Laplace and Helmholtz equations are to some extent similar in structure: on the basis of the classification existing in mathematical physics, both equations are elliptic equations. They are particular cases of the general Poisson equation. The use of the numerical - analytical method for the Helmholtz equation was considered [5] for cases of moderate values of the wave number. In this paper, we consider the behavior of this circuit for large wave number.

The aim of this work is to study and develop the application of the numerical-analytical method for boundary-value problems for which a rapidly oscillating function is given at the boundary. The study is based on the Helmholtz equation. The numerical solution of the Laplace equation is constructed in the same way as the Helmholtz equation, but using some modification.

## 1. Application of the method for the Helmholtz equation. The case of a rapidly oscillating function at the boundary

In [6], a scheme for applying the numerical-analytical method for the Helmholtz equation was considered using the example of a two-dimensional domain $[a,b]\times[c,d]$:

$$\Delta f + k^2 f = 0 \tag{2}$$

$$f_x(x,y) = 0,\ \ x = 0,\ \ f_x(x,y) = 0,\ \ x = a; \tag{3}$$

$$-f_y(x,y) = V_0,\ \ y = 0,\ \ -f_y(x,y) = 0,\ \ y = b\,. \tag{4}$$

In this case, the sound potential, following the idea of the method, is represented in the form:

$$f(y)=f(y_0)+f_y(y_0)(y-y_0)+\frac{1}{2!}f_{yy}(y_0)(y-y_0)^2+$$
$$+\frac{1}{3!}f_{yyy}(y_0)(y-y_0)^3+o((y-y_0)^3), \quad (5)$$

where $f(y_0),\ f_y(y_0),\ f_{yy}(y_0),\ f_{yyy}(y_0)$ unknown row expansion coefficients. We denote

$$x_1=f(y_0),\quad x_2=f_y(y_0),\ x_3=f_{yy}(y_0),\ x_4=f_{yyy}(y_0).$$

Applying a 4-point scheme, we obtain the following recurrence relations [5]:

$$x_3=\frac{(-0.5f(0)+2f(\Delta)-2.5f(2\Delta))\cdot k^2}{0.5\Delta^2k^2+1} \quad (6)$$

$$x_4=\frac{(2f(0)-3f(\Delta)-3\Delta^2x_3)\cdot k^2-x_3}{-5\Delta^3k^2} \quad (7)$$

$$x_2=-\frac{f(0)-f(\Delta)-2.5x_3\Delta^2+\frac{19}{6}\Delta^3x_4}{\Delta} \quad (8)$$

$$x_1=f(0)+3\Delta\cdot x_2-4.5\Delta^2\cdot x_3+4.5\Delta^3\cdot x_4. \quad (9)$$

Approximations of the third order are quite enough for this method so that the calculated interval $[0;1]$ , broken with a step $\Delta=0,02$, already gives us order accuracy $10^{-3}-10^{-4}$. However, the range of wave numbers $k$ was $0\le k\le 8$. In this case, the calculation was performed for a little more than one wavelength. How will the method behave if 3 or 5 wavelengths are placed on the calculation interval? In this problem, this corresponds to order wavenumbers $k=20,30$. The numerical calculation showed that the step $\Delta=0,02$ here will be too large: if you do not change it, then, starting with $y\approx 0,3$, a solution shift is observed according to this numerical method in comparison with the analytical solution by the Fourier method.

If we grind the step $\Delta=1/120$, then for $k=20$ (a little more than 3 wavelengths) we get results that coincide with the analytical solution with good accuracy. In Fig. 1, the dashed curve corresponds to the solution according to the numerical analytical method, the solid line is the exact solution according to the Fourier method. For $k=30$ (about 5 wavelengths) the step $\Delta=1/120$ must be ground to $\Delta=1/180$. Nevertheless, we see that there is a slight deviation at the end of the calculation interval of the numerical solution according to the method from the analytical solution.

In terms of accuracy, the solution here is slightly inferior to the solution for small values $k$ [5].

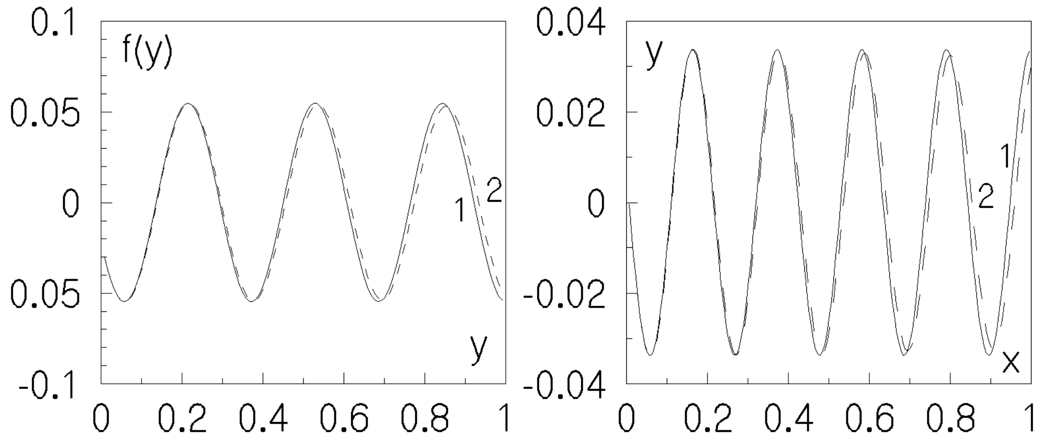


Fig.1 Sound pressure potential for wave numbers $k$ =20, 30.

**2. Application of the method to the Laplace equation**

Let we have a rectangular region $[-a,a]\times[-h,h]$ symmetric with respect to the origin in which the Laplace equation (1) is solved. At the boundary of the region, the following boundary conditions are specified:

$$f=0, \quad x=\pm a, \tag{10}$$

$$f=A_0\cos\frac{(2n+1)}{2a}\pi x, \ \ y=\pm h, \tag{11}$$

where $A_0$ – is some constant.

If we assume that at the boundary $y=\pm h$ the function $f$ depends only on $x$, then the Laplace equation takes the form:

$$f_{xx}=0. \tag{12}$$

And this means that $f=C_1x+C_2$, where $C_1$, $C_2$ are some integration constants of equation (12). But this contradicts the fact that the function $f=A_0\cdot\cos\frac{(2n+1)\pi}{2a}x$ is at the given boundary. Therefore, the direct application of the method by analogy with the Helmholtz equation does not work.

It is possible to solve the arisen problem if we know what structure the desired solution should be. Based on the Fourier method, it is easy to obtain:

$$f(x,y)=\sum_{n=0}^{\infty}A_n\cos\lambda_n x\cdot(C_n e^{-\lambda_n y}+D_n e^{\lambda_n y}), \tag{13}$$

where $\lambda_n=\frac{2n+1}{2a}\pi$.

However, let us pay an attention to expression (13). If we differentiate both sides of (13) by a variable $y$, we obtain:

$$f_{yy}=\lambda_n^2 f. \tag{14}$$

Let us return to the numerical-analytical method. Replace in the Laplace equation $f_{yy}$ by $\lambda_n^2 f$:

$$f_{xx} + \lambda_n^2 f = 0. \qquad (15)$$

In appearance, this equation exactly matches the Helmholtz equation, with one exception, which is $k^2$ instead of $\lambda_n^2$ here. Therefore, we can further use the scheme for applying the numerical-analytical method for the Helmholtz equation (2), replacing $k^2$ by $\lambda_n^2$:

$$x_3 = \frac{(-0.5f(0) + 2f(\Delta) - 2.5f(2\Delta)) \cdot \lambda_n{}^2}{0.5\Delta^2\lambda_n{}^2 + 1}, \qquad (16)$$

$$x_4 = \frac{(2f(0) - 3f(\Delta) - 3\Delta^2 x_3) \cdot \lambda_n{}^2 - x_3}{-5\Delta^3\lambda_n{}^2}, \qquad (17)$$

$$x_2 = -\frac{f(0) - f(\Delta) - 2.5x_3\Delta^2 + \frac{19}{6}\Delta^3 x_4}{\Delta}, \qquad (18)$$

$$x_1 = f(0) + 3\Delta \cdot x_2 - 4.5\Delta^2 \cdot x_3 + 4.5\Delta^3 \cdot x_4. \qquad (19)$$

The numerical calculation showed the same degree of convergence of the solution to the analytical solution (Fig. 2). The use of substitution (14) allows us to generalize the scheme of using the numerical-analytical method for the Laplace equation and a function $\psi$ arbitrarily defined on the boundary.

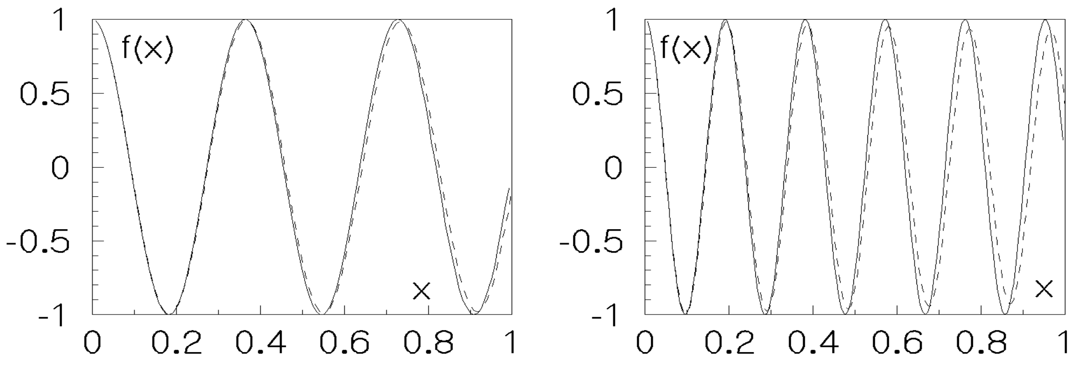


Fig. 2. The flow potential for $n = 5$, 10

**3. A generalization of the application of the numerical analytical method for the Laplace equation**

Let not one mode (11), but some arbitrary function $\psi$ be given on the boundary of the region $[-a, a] \times [-h, h]$:

$$f(x, \pm h) = \psi, \qquad (20)$$

which satisfies the Laplace equation. The boundary condition (10) remains the same. Applying the scheme of the Fourier method, we obtain (13). The set of eigenvalues $\lambda_n$ depends on the type of boundary condition. For a fixed value $y = \pm h$ at the boundary, the two-dimensional function $f(x, y)$ is transformed into the usual trigonometric series:

$$\psi(x) = \sum_{n=0}^{\infty} A_n^* \cos \lambda_n x = \psi_0 + \psi_1 + ... + \psi_n, \quad A_n^* = C_n e^{-\lambda_n(\pm h)} + D_n e^{\lambda_n(\pm h)}. \tag{21}$$

Expression (21) is a trigonometric series composed of eigenfunctions obtained based on the solution of the Sturm-Liouville problem. It is not difficult to carry out rationing in such a way as to obtain an orthonormal closed system of functions. Therefore, according to the Riesz-Fisher and Steklov-Parseval theorems [6], expression (22) is a unique representation of the function in the form of a Fourier series in terms of the eigenfunctions of the Sturm-Liouville problem.

Now back to the function $f$. From the theory of linear differential equations it is known that a linear combination of individual solutions of a linear differential equation is also a solution to this differential equation. The Laplace equation is linear, therefore, if we present the desired solution in the form of a superposition of $n$ solutions $f_i$:

$$f = \sum_{i=0}^{n} f_i, \tag{22}$$

then this superposition for each $i$ will also be a solution to the Laplace equation:

$$\Delta f_i = 0. \tag{23}$$

This statement allows us to break down the original task for the function $f$ on the $n+1$ task:

$$\begin{aligned} &\Delta f_i = 0, \\ &f_i = 0, \quad x = \pm a, \\ &f_i = \psi_i, \; y = \pm h, \; i = \overline{0, n}. \end{aligned} \tag{24}$$

It is easy to see that the obtained boundary-value problem (24) for each fixed value $i$ coincides with the problem described above for a strongly oscillating function $f$, for which the scheme for applying the numerical-analytical method has already been considered.

*Comment.* Such a simple implementation of the scheme of the numerical analytical method is possible only for the cases of linear equations considered in the paper. If we take the quasi-linear equation [7], then the analytical representation of the solution in the a recursive form will not work: the nonlinear system of equations that is formed as a result of applying the numerical-analytical method has to be solved numerically.

**4. An example of the application of the method for quasi-linear equations**

This section provides a solution to the problem of transonic flow around a plane wing profile. The equation describing the flow around a wing profile is a quasilinear equation:

$$[1-\frac{1}{M_1^2}+\varepsilon\cdot(\gamma+1)f_\xi]f_{\xi\xi}-\frac{\lambda^2}{M_1^2}f_{\eta\eta}=0, \quad (25)$$

in dimensionless coordinates $\xi = x/c,\ \eta = \lambda y$, where $x,\ y$ – the Cartesian coordinates along and across the section of the blade, respectively. The coordinate $\eta$ displays the surface shape of the cross section of the blade. Taking into account accepted notation, the boundary condition on the surface of the blade is written in the form:

$$\eta = g(\xi),\ 0<\xi<1; f_\eta = \delta g_\xi,$$

We turn to the application of the method. We believe that $f \in C^2([0;1]\times[0;1])$. Then it can be represented as a Taylor series in a neighborhood of an arbitrary point $(\xi_i,\eta_i)$:

$$\begin{gathered} f(\xi_i,\eta_i)=f(\xi_0,\eta_0)+f_\xi(\xi_0,\eta_0)(\xi_i-\xi_0)+f_\eta(\xi_0,\eta_0)(\eta_i-\eta_0)+\\ +\frac{1}{2}[f_{\xi\xi}(\xi_0,\eta_0)(\xi_i-\xi_0)^2+2f_{\xi\eta}(\xi_0,\eta_0)(\xi_i-\xi_0)(\eta_i-\eta_0)+f_{\eta\eta}(\xi_0,\eta_0)(\eta_i-\eta_0)^2]+ \\ +\ o(\max\{(\xi_i-\xi_0)^2,(\xi_i-\xi_0)(\eta_i-\eta_0),(\eta_i-\eta_0)^2\}),\ i=1,5\,. \end{gathered} \quad (26)$$

Here the point $(\xi_0,\eta_0)$ is some fixed point of profile, close to $(\xi_i,\eta_i)$ (Fig.3).

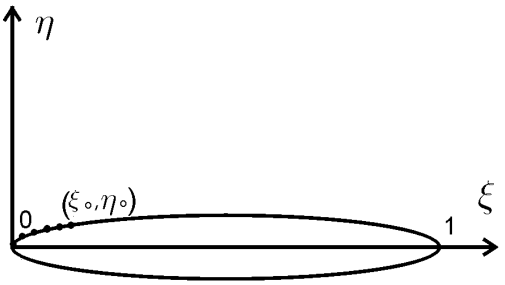


Fig. 3. Calculation pattern on the surface of the wing profile

Calculations will be carried out along the upper edge of the wing. We select five points at the beginning of the profile on the left with coordinates $(\xi_i,\eta_i)$, $i=1,5$. We regard that $(\xi_0,\eta_0)$ is the six point, that is situated on the upper boundary of wing just after the first five points. System (4) consists of five equations with six unknowns $f(\xi_0,\eta_0), f_\xi(\xi_0,\eta_0)$, $f(\xi_0,\eta_0), f_\xi(\xi_0,\eta_0), f_{\xi\eta}(\xi_0,\eta_0), f_{\eta\eta}(\xi_0,\eta_0)$.

As $(\xi_0,\eta_0)$ is an arbitrary point, then equation (1) should be performed for it automatically, i.e.

$$[1-\frac{1}{M_1^2}+\varepsilon\cdot(\gamma+1)f_\xi(\xi_0,\eta_0)]f_{\xi\xi}(\xi_0,\eta_0)-\frac{\lambda^2}{M_1^2}f_{\eta\eta}(\xi_0,\eta_0)=0 \quad (27)$$

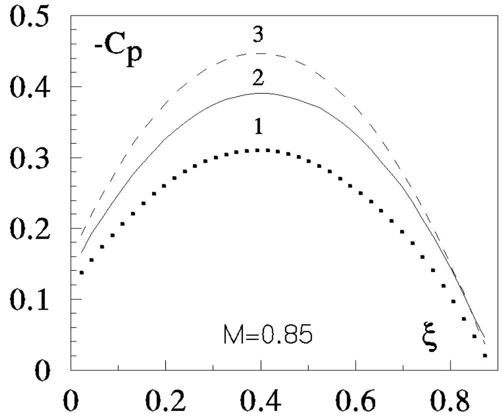


Fig. 4a

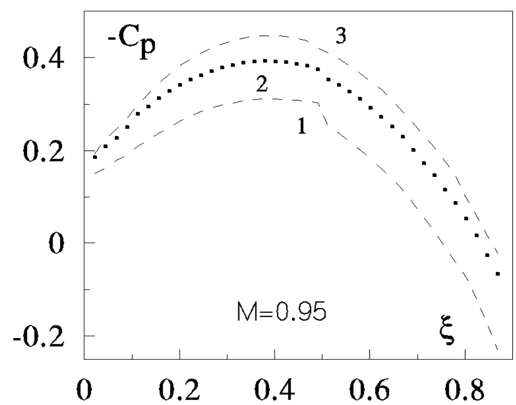


Fig. 4b

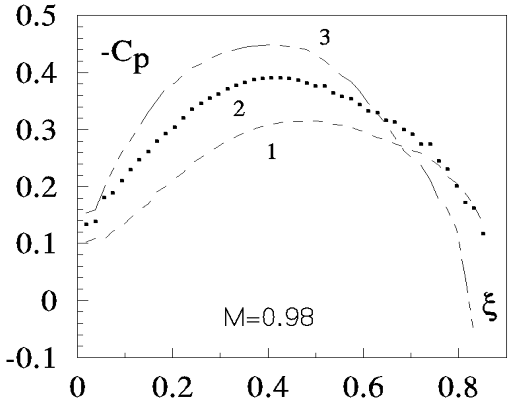


Fig. 5a

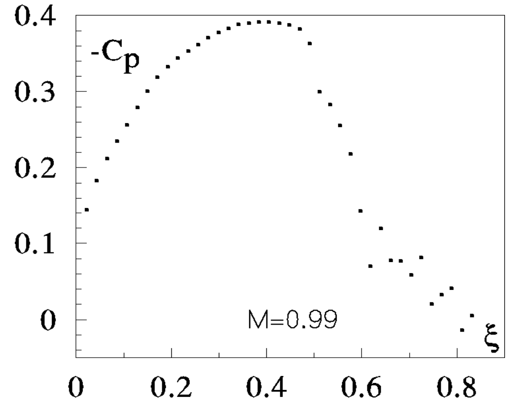


Fig. 5b

The system of equations (26)-(27) is closed relative to the indicated unknowns if we omit the second-order quantities of smallness. The data of calculating the pressure coefficient on the surface of the blade are shown in Fig. 4-5.

Close agreement is obtained with the calculated data of [8] in the case of a helicopter rotor blade remote from the end of the flow around the blade. At $M > 0{,}9$ the nucleation of a weak shock wave is observed (Fig. 3), which is realized in the form of a small jump literally immediately after $\xi = 0.5$.

As you approach $M = 1$ ($M = 0.98; 0.99$) it is noticeable instable behavior of $C_p$ (fig.5). This is the cause of the instability of the flow itself. Since equation (25) changes type from elliptic to hyperbolic on a shock wave, it affects the numerical algorithm, where the series expansion was assumed to be continuous. Here, the method reveals an interesting regularity of the flow: the so-called “spurious vorticity”. In this zone, the method works stably even with unstable flow behavior.

The study of shock wave arising is very import in safety of flight air vehicles. So the numerical-analytical method can be used as an information technology for environment.

**Conclusions**

1. The paper considers the algorithm for applying the numerical-analytical method for the Laplace and Helmholtz equations for the case of strongly oscillating functions.

2. A comparison is made of the numerical convergence of the method with an analytical solution.

3. The features of setting the grid domain, the choice of the calculation step for the optimal use of the method scheme were studied.

4. The behavior of the helicopter blade in the mode of nucleation of a shock wave, the occurrence of flutter of the blade is studied. The calculation data can be used in the design of information flight control systems for aircraft, helicopters in order to prevent air crashes - local environmental incidents.

5. The numerical-analytical method proposed in the paper allows for low computational costs to solve numerically the problems for the Laplace equation and the acoustic equations. The results of the research conducted can be used as new information technology to address the environmental security challenges of water and air resources.

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**ЧИСЕЛЬНО-АНАЛІТИЧНИЙ МЕТОД РОЗВ'ЯЗАННЯ ЗАДАЧ ЗАХИСТУ ОТОЧУЮЧОГО СЕРЕДОВИЩА**

***Анотація.*** *У задачах з моделювання забруднення води, атмосфери Землі часто використовують потенціальні моделі течії. Нещодавно значну увагу стали придіяти захисту метрополісів від акустичних шумових забруднень. Часто, не фокусуючи увагу на локальних рисах течії, для даного моделювання використовується рівняння Лапласа, яке описує*

*потенціальний рух рідини. Акустичні задачі моделюються на основі рівняння Гельмгольца. У даній роботі розглянуто особливості використання чисельно-аналітичного методу для рівнянь Лапласа та Гельмгольца.. Звуковий потенціал є швидко осцилюючою функцією, заданою на границі розрахункової області. До того ж, представлено особливості чисельно-аналітичного методу для рівняння Лапласа з використанням розвинення граничної умови в ряд Тейлора за вланими функціями задачі Штурма-Ліувілля. Не зважаючи на те, що дані, присутні у даній роботі, отримані для канонічних областей, схема методумає на увазі його використання для довільної криволінійної границі області. Чисельно-аналітичний метод, запропонований в даній роботі, дозволяє малими розрахунковими затратами чисельно розв'язувати задачі для рівняння Лапласа і акустичні рівняння. Результати даних досліджень можуть бути використані у якості нових інформаційних технологій для захисту оточуючого середовища.*

***Мета статті.*** *Метою статті є вивчення та розвиток застосування чисельно-аналітичного методу для граничних задач зі швидко-осцилюючими функціями, які задані на границі області.*

***Висновки.*** *У даній роботі запропоновано алгоритм застосування чисельно-аналітичного методу для рівнянь Лапласа та Гельмгольца у випдку швидко осцилюючих функцій на границі. Виконано порівняння збіжності чисельного розв'язку задачи з аналітичним. Встановлено оптимальну кількість розбиття сітки розрахункової області для досягнення оптимальної збіжності чисельного розв'язку до аналітичного. Розв'язано задачу виникнення звукового поля для критичного діапазону обтікання лопаті, де виникають ударні хвилі, що можуть спричинити руйнування лопаті. Це дослідження може стати у нагоді екологічної безпеки польотів на гелікоптерах. Як показали дані розрахунку, чисельно-аналітичний метод здатний з порівняно малими затратами розв'язати граничні задачі для рівняння Лапласа та Гельмгольца. Результати даних досліджень спрямовані на розв'язання задач екологічної безпеки водних та повітряних ресурсів Землі.*

***Ключові слова:*** *комп'ютерне моделювання, чисельний метод, екологічний моніторінг оточуючого середовища.*

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## ЗАСТОСУВАННЯ ІНСТРУМЕНТАРІЮ ІНТЕГРАЛЬНОГО ЧИСЛЕННЯ ДО РОЗВ'ЯЗУВАННЯ ЗАДАЧ ГЕОМЕТРИЧНОГО ТА МЕХАНІЧНОГО ЗМІСТУ З МАТЕМАТИЧНОГО АНАЛІЗУ

*У статті розглянуто семіотичні особливості вивчення математичних формул з курсу математичного аналізу, зокрема інтегрального числення; запропоновано закріплювати вивчені формули за трьома етапами (початковий, середній, заключний); показано застосування кожного*