СЕКЦІЯ «ПРИКЛАДНА МАТЕМАТИКА»

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NON SYMPLIFIED EQUATIONS SYSTEM OF k-ε TURBULENCE MODEL IN CYLINDRICAL COORDINATES

Laminar Navier-Stokes equations in cylindrical coordinates are described in most books on theoretical hydrodynamics [1]-[3]. *Reynolds Equations can easy be found in the literature also. But the full equations system of k-ε turbulence model in cylindrical coordinates is very difficult to find. They are usually described in papers only in simplified form. In given paper non simplified system of k-ε turbulence model equations in cylindrical coordinates is derived*

Key words: *k-ε turbulence model, Reynolds equation, cylindrical coordinates.*

1. Reynolds equation in cylindrical coordinates.

Navier-Stokes equations in stresses in cylindrical coordinates have a form

$$\frac{\partial U_r}{\partial t}+U_r\frac{\partial U_r}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_r}{\partial \varphi}+U_z\frac{\partial U_r}{\partial z}-\frac{U_\varphi^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}-\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\varphi}}{\partial \varphi}-\frac{\tau_{\varphi\varphi}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right) \quad (1)$$

$$\frac{\partial U_\varphi}{\partial t}+U_r\frac{\partial U_\varphi}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_\varphi}{\partial \varphi}+U_z\frac{\partial U_\varphi}{\partial z}+\frac{U_\varphi U_r}{r}=-\frac{1}{r}\frac{1}{\rho}\frac{\partial P}{\partial \varphi}-\frac{1}{\rho}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\varphi}\right)+\frac{1}{r}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi}+\frac{\partial \tau_{\varphi z}}{\partial z}\right) \quad (2)$$

$$\frac{\partial U_z}{\partial t}+U_r\frac{\partial U_z}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_z}{\partial \varphi}+U_z\frac{\partial U_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}-\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\varphi z}}{\partial \varphi}+\frac{\partial \tau_{zz}}{\partial z}\right) \quad (3)$$

$$\frac{1}{r}\frac{\partial rU_r}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi}+\frac{\partial U_z}{\partial z}=\frac{\partial U_r}{\partial r}+\frac{U_r}{r}+\frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi}+\frac{\partial U_z}{\partial z}=0 \quad (4)$$

Friction stresses τ are given by the formulas

$$\tau_{rr}=-2\mu\frac{\partial U_r}{\partial r};\quad \tau_{\varphi\varphi}=-2\mu\left(\frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi}+\frac{U_r}{r}\right);\quad \tau_{zz}=-2\mu\frac{\partial U_z}{\partial z}; \quad (5)$$

$$\tau_{r\varphi}=\tau_{\varphi r}=-\mu\left(\frac{1}{r}\frac{\partial U_r}{\partial \varphi}+\frac{\partial U_\varphi}{\partial r}-\frac{U_\varphi}{r}\right);\quad \tau_{rz}=\tau_{zr}=-\mu\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right);\quad \tau_{\varphi z}=\tau_{z\varphi}=-\mu\left(\frac{1}{r}\frac{\partial U_z}{\partial \varphi}+\frac{\partial U_\varphi}{\partial z}\right); \quad (6)$$

After substituting of (5)-(6) into (1)-(3) right side of (1)-(3) can be transformed to the form

$$\frac{\partial}{\partial r}\frac{1}{r}\mu\frac{\partial rU_r}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\mu\frac{\partial U_r}{\partial\varphi}-2\mu\frac{1}{r^2}\frac{\partial U_\varphi}{\partial\varphi}+\frac{\partial}{\partial z}\mu\frac{\partial U_r}{\partial z}+\frac{\partial}{\partial r}\mu\left(\frac{1}{r}\frac{\partial rU_r}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{\partial U_r}{\partial z}\right)$$

$$\left[\frac{\partial}{\partial r}\mu\frac{\partial U_\varphi}{\partial r}+\frac{1}{r}\mu\frac{\partial U_\varphi}{\partial r}-\frac{\mu U_\varphi}{r^2}\right]+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\mu\frac{\partial U_\varphi}{\partial\varphi}+2\frac{1}{r^2}\frac{\partial\mu U_r}{\partial\varphi}+\frac{\partial}{\partial z}\mu\frac{\partial U_\varphi}{\partial z}+\frac{1}{r}\frac{\partial}{\partial\varphi}\mu\left(\frac{1}{r}\frac{\partial rU_r}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{\partial U_z}{\partial z}\right)$$

$$\left[\frac{\partial}{\partial r}\mu\frac{\partial U_z}{\partial r}+\frac{1}{r}\mu\frac{\partial U_z}{\partial r}\right]+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\mu\frac{\partial U_z}{\partial\varphi}+\frac{\partial}{\partial z}\mu\frac{\partial U_z}{\partial z}+\frac{\partial}{\partial z}\mu\left(\frac{1}{r}\frac{\partial rU_r}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{\partial U_z}{\partial z}\right)$$

Note, than transformation is correct only if $\mu = const$ **.**

In round brackets the continuity equation is placed, so these brackets can be omitted. The terms in the square brackets are rewritten as single term.

$$\begin{aligned}&\frac{\partial U_r}{\partial t}+U_r\frac{\partial U_r}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_r}{\partial\varphi}+U_z\frac{\partial U_r}{\partial z}-\frac{U_\varphi^2}{r}=\\&=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\frac{1}{\rho}\left(\frac{\partial}{\partial r}\frac{1}{r}\mu\frac{\partial rU_r}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\mu\frac{\partial U_r}{\partial\varphi}-2\mu\frac{1}{r^2}\frac{\partial U_\varphi}{\partial\varphi}+\frac{\partial}{\partial z}\mu\frac{\partial U_r}{\partial z}\right)\end{aligned}\tag{7}$$

$$\begin{aligned}&\frac{\partial U_\varphi}{\partial t}+U_r\frac{\partial U_\varphi}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_\varphi}{\partial\varphi}+U_z\frac{\partial U_\varphi}{\partial z}+\frac{U_rU_\varphi}{r}=\\&=-\frac{1}{r}\frac{1}{\rho}\frac{\partial P}{\partial\varphi}+\frac{1}{\rho}\left(\frac{\partial}{\partial r}\frac{1}{r}\mu\frac{\partial rU_\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\mu\frac{\partial U_\varphi}{\partial\varphi}+2\frac{1}{r^2}\mu\frac{\partial U_r}{\partial\varphi}+\frac{\partial}{\partial z}\mu\frac{\partial U_\varphi}{\partial z}\right)\end{aligned}\tag{8}$$

$$\begin{aligned}&\frac{\partial U_z}{\partial t}+U_r\frac{\partial U_z}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_z}{\partial\varphi}+U_z\frac{\partial U_z}{\partial z}=\\&=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\mu r\frac{\partial U_z}{\partial r}+\frac{\partial}{\partial z}\mu\frac{\partial U_z}{\partial z}+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\mu\frac{\partial U_z}{\partial\varphi}\right)\end{aligned}\tag{9}$$

Now consider as velocity fluctuations forces on flow. For this we rewrite left side of (7)-(9) using continuity equation in conservative form.

$$\begin{aligned}&U_r\frac{\partial U_r}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_r}{\partial\varphi}+U_z\frac{\partial U_r}{\partial z}-\frac{U_\varphi^2}{r}+U_r\left(\frac{\partial U_r}{\partial r}+\frac{\partial U_z}{\partial z}+\frac{1}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{U_r}{r}\right)=\\&=\frac{1}{r}\frac{\partial rU_rU_r}{\partial r}+\frac{1}{r}\frac{\partial U_rU_\varphi}{\partial\varphi}+\frac{\partial U_rU_z}{\partial z}-\frac{U_\varphi^2}{r}\end{aligned}\tag{10}$$

$$\begin{aligned}&U_r\frac{\partial U_\varphi}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{U_rU_\varphi}{r}+U_z\frac{\partial U_\varphi}{\partial z}+U_\varphi\left(\frac{\partial U_r}{\partial r}+\frac{\partial U_z}{\partial z}+\frac{1}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{U_r}{r}\right)=\\&=\frac{1}{r^2}\frac{\partial r^2U_rU_\varphi}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi U_\varphi}{\partial\varphi}+\frac{\partial U_\varphi U_z}{\partial z}\end{aligned}\tag{11}$$

$$\begin{aligned}
&U_r\frac{\partial U_z}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_z}{\partial \varphi}+U_z\frac{\partial U_z}{\partial z}+U_z\left(\frac{\partial U_r}{\partial r}+\frac{\partial U_z}{\partial z}+\frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi}+\frac{U_r}{r}\right)=\\
&=\frac{1}{r}\frac{\partial rU_rU_z}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi U_z}{\partial \varphi}+\frac{\partial U_zU_z}{\partial z}
\end{aligned} \tag{12}$$

Now we decomposed instantaneous quantities into sum of time-averaged and fluctuating parts, substitute this sum into (10)-(12) and average result. After this expressions for averaged velocities rewrite in original, non conservative form.

$$\begin{aligned}
&U_r\frac{\partial U_z}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_z}{\partial \varphi}+U_z\frac{\partial U_z}{\partial z}=\frac{1}{r}\frac{\partial rU_rU_z}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi U_z}{\partial \varphi}+\frac{\partial U_zU_z}{\partial z}=\\
&=\overline{U_r}\frac{\partial \overline{U_z}}{\partial r}+\frac{\overline{U_\varphi}}{r}\frac{\partial \overline{U_z}}{\partial \varphi}+\overline{U_z}\frac{\partial \overline{U_z}}{\partial z}+\frac{1}{r}\frac{\partial r\overline{u_ru_z}}{\partial r}+\frac{1}{r}\frac{\partial \overline{u_\varphi u_z}}{\partial \varphi}+\frac{\partial \overline{u_zu_z}}{\partial z}
\end{aligned} \tag{13}$$

$$\begin{aligned}
&U_r\frac{\partial U_\varphi}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_\varphi}{\partial \varphi}+\frac{U_rU_\varphi}{r}+U_z\frac{\partial U_\varphi}{\partial z}=\frac{1}{r^2}\frac{\partial r^2U_rU_\varphi}{\partial r}+\frac{1}{r}\frac{\partial U_\varphi U_\varphi}{\partial \varphi}+\frac{\partial U_\varphi U_z}{\partial z}=\\
&=\overline{U_r}\frac{\partial \overline{U_\varphi}}{\partial r}+\frac{\overline{U_\varphi}}{r}\frac{\partial \overline{U_\varphi}}{\partial \varphi}+\frac{\overline{U_r}\,\overline{U_\varphi}}{r}+\overline{U_z}\frac{\partial \overline{U_\varphi}}{\partial z}+\frac{1}{r^2}\frac{\partial r^2\overline{u_ru_\varphi}}{\partial r}+\frac{1}{r}\frac{\partial \overline{u_\varphi u_\varphi}}{\partial \varphi}+\frac{\partial \overline{u_\varphi u_z}}{\partial z}
\end{aligned} \tag{14}$$

$$\begin{aligned}
&U_r\frac{\partial U_r}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_r}{\partial \varphi}+U_z\frac{\partial U_r}{\partial z}-\frac{U_\varphi^2}{r}=\frac{1}{r}\frac{\partial rU_rU_r}{\partial r}+\frac{1}{r}\frac{\partial U_rU_\varphi}{\partial \varphi}+\frac{\partial U_rU_z}{\partial z}-\frac{U_\varphi^2}{r}=\\
&=\frac{\overline{U_\varphi}}{r}\frac{\partial \overline{U_r}}{\partial \varphi}+\overline{U_r}\frac{\partial \overline{U_r}}{\partial r}+\overline{U_z}\frac{\partial \overline{U_r}}{\partial z}-\frac{\overline{U_\varphi^2}}{r}+\frac{1}{r}\frac{\partial r\overline{u_ru_r}}{\partial r}+\frac{1}{r}\frac{\partial \overline{u_ru_\varphi}}{\partial \varphi}+\frac{\partial \overline{u_ru_z}}{\partial z}-\frac{\overline{u_\varphi u_\varphi}}{r}
\end{aligned} \tag{15}$$

Now substitute (13)-(15) into (1)-(3). After simple transformation we obtain:

$$\begin{aligned}
&\frac{\partial \overline{U_r}}{\partial t}+\overline{U_r}\frac{\partial \overline{U_r}}{\partial r}+\frac{\overline{U_\varphi}}{r}\frac{\partial \overline{U_r}}{\partial \varphi}+\overline{U_z}\frac{\partial \overline{U_r}}{\partial z}-\frac{\overline{U_\varphi^2}}{r}\\
&=-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial r}+\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\tau_{rr}-\overline{u_ru_r}\right)\right)+\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\tau_{r\varphi}-\overline{u_ru_\varphi}\right)-\frac{\left(\tau_{\varphi\varphi}-\overline{u_\varphi u_\varphi}\right)}{r}+\frac{\partial}{\partial z}\left(\tau_{rz}-\overline{u_ru_z}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial U_\varphi}{\partial t}+U_r\frac{\partial U_\varphi}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_\varphi}{\partial \varphi}+U_z\frac{\partial U_\varphi}{\partial z}+\frac{U_\varphi U_r}{r}\\
&=-\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \varphi}+\frac{1}{\rho}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\tau_{r\varphi}-\overline{u_ru_\varphi}\right)\right)+\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\tau_{\varphi\varphi}-\overline{u_\varphi u_\varphi}\right)+\frac{\partial}{\partial z}\left(\tau_{\varphi z}-\overline{u_\varphi u_z}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial U_z}{\partial t}+U_r\frac{\partial U_z}{\partial r}+\frac{U_\varphi}{r}\frac{\partial U_z}{\partial \varphi}+U_z\frac{\partial U_z}{\partial z}\\
&=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\tau_{rz}-\overline{u_ru_z}\right)\right)+\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\tau_{\varphi z}-\overline{u_\varphi u_z}\right)+\frac{\partial}{\partial z}\left(\tau_{zz}-\overline{u_zu_z}\right)\right)
\end{aligned}$$

Following Boussinesq hypothesis we introduce turbulent viscosity as:

$$-\overline{u_r u_r} = 2\mu \frac{\partial U_r}{\partial r}; \quad -\overline{u_\varphi u_\varphi} = 2\mu_t \left(\frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi} + \frac{U_r}{r} \right); \quad -\overline{u_z u_z} = 2\mu_t \frac{\partial U_z}{\partial z};$$

$$-\overline{u_r u_\varphi} = \mu_t \left(\frac{1}{r}\frac{\partial U_r}{\partial \varphi} + \frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r} \right); \quad -\overline{u_r u_z} = \mu_t \left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} \right); \quad -\overline{u_\varphi u_z} = \mu_t \left(\frac{1}{r}\frac{\partial U_z}{\partial \varphi} + \frac{\partial U_\varphi}{\partial z} \right).$$

Then turbulent Navier-Stokes or Reynolds equations can be written in the form:

$$\begin{aligned} &\frac{\partial U_r}{\partial t} + U_r \frac{\partial U_r}{\partial r} + \frac{U_\varphi}{r}\frac{\partial U_r}{\partial \varphi} + U_z \frac{\partial U_r}{\partial z} - \frac{U_\varphi^2}{r} = \\ &= -\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{1}{\rho}\left(\frac{\partial}{\partial r}\frac{1}{r}(\mu+\mu_t)\frac{\partial r U_r}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \varphi}(\mu+\mu_t)\frac{\partial U_r}{\partial \varphi} - 2(\mu+\mu_t)\frac{1}{r^2}\frac{\partial U_\varphi}{\partial \varphi} + \frac{\partial}{\partial z}(\mu+\mu_t)\frac{\partial U_r}{\partial z} \right) \end{aligned} \tag{16}$$

$$\begin{aligned} &\frac{\partial U_\varphi}{\partial t} + U_r \frac{\partial U_\varphi}{\partial r} + \frac{U_\varphi}{r}\frac{\partial U_\varphi}{\partial \varphi} + U_z \frac{\partial U_\varphi}{\partial z} + \frac{U_r U_\varphi}{r} = \\ &= -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \varphi} + \frac{1}{\rho}\left(\frac{\partial}{\partial r}\frac{1}{r}\left((\mu+\mu_t)\frac{\partial r U_\varphi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}(\mu+\mu_t)\frac{\partial U_\varphi}{\partial \varphi} + 2\frac{1}{r^2}(\mu+\mu_t)\frac{\partial U_r}{\partial \varphi} + \frac{\partial}{\partial z}(\mu+\mu_t)\frac{\partial U_\varphi}{\partial z} \right) \end{aligned} \tag{17}$$

$$\begin{aligned} &\frac{\partial U_z}{\partial t} + U_r \frac{\partial U_z}{\partial r} + \frac{U_\varphi}{r}\frac{\partial U_z}{\partial \varphi} + U_z \frac{\partial U_z}{\partial z} = \\ &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}(\mu+\mu_t) r \frac{\partial U_z}{\partial r} + \frac{\partial}{\partial z}(\mu+\mu_t)\frac{\partial U_z}{\partial z} + \frac{1}{r^2}\frac{\partial}{\partial \varphi}(\mu+\mu_t)\frac{\partial U_z}{\partial \varphi} \right) \end{aligned} \tag{18}$$

2. Turbulent kinetic energy transfer, convective and pressure terms.

Now we decomposed instantaneous velocities and pressure as sum of time-averaged and fluctuating parts $U_r = \overline{U_r} + u_r$, $U_\varphi = \overline{U_\varphi} + u_\varphi$, $U_z = \overline{U_z} + u_z$ $\overline{P} = P + p$, substitute this sum into (1)-(3) and average result. Diffusion terms will be transformed separately.

$$\begin{aligned} &u_\varphi \left(\frac{\partial(\overline{U_\varphi}+u_\varphi)}{\partial t} + (\overline{U_r}+u_r)\frac{\partial(\overline{U_\varphi}+u_\varphi)}{\partial r} + \frac{(\overline{U_\varphi}+u_\varphi)}{r}\frac{\partial(\overline{U_\varphi}+u_\varphi)}{\partial \varphi} + (\overline{U_z}+u_z)\frac{\partial(\overline{U_\varphi}+u_\varphi)}{\partial z} + \frac{(\overline{U_r}+u_r)(\overline{U_\varphi}+u_\varphi)}{r} + \right. \\ &\left. + \frac{1}{\rho}\frac{1}{r}\frac{\partial(\overline{P}+p)}{\partial \varphi} \right) = \left(\frac{\partial(0.5u_\varphi^2)}{\partial t} + \overline{U_r}\frac{\partial(0.5u_\varphi^2)}{\partial r} + \frac{\overline{U_\varphi}}{r}\frac{\partial(0.5u_\varphi^2)}{\partial \varphi} + \overline{U_z}\frac{\partial(0.5u_\varphi^2)}{\partial z} \right) + \left(u_\varphi u_r \frac{\partial \overline{U_\varphi}}{\partial r} + u_\varphi u_\varphi \frac{1}{r}\frac{\partial \overline{U_\varphi}}{\partial \varphi} + \right. \\ &\left. + u_\varphi u_z \frac{\partial \overline{U_\varphi}}{\partial z} \right) = \left(u_\varphi u_r \frac{\partial u_\varphi}{\partial r} + u_\varphi u_\varphi \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + u_\varphi u_z \frac{\partial u_\varphi}{\partial z} \right) + \left(u_\varphi u_\varphi \frac{\overline{U_r}}{r} + u_\varphi u_r \frac{\overline{U_\varphi}}{r} + \frac{u_r u_\varphi u_\varphi}{r} \right) + \frac{1}{\rho}\frac{1}{r}u_\varphi \frac{\partial p}{\partial \varphi} \end{aligned} \tag{19}$$

$$u_r\left(\frac{\partial(\overline{U_r}+u_r)}{\partial t}+(\overline{U_r}+u_r)\frac{\partial(\overline{U_r}+u_r)}{\partial r}+\frac{(\overline{U_\varphi}+u_\varphi)}{r}\frac{\partial(\overline{U_r}+u_r)}{\partial\varphi}+(\overline{U_z}+u_z)\frac{\partial(\overline{U_r}+u_r)}{\partial z}-\frac{(\overline{U_\varphi}+u_\varphi)^2}{r}+\frac{1}{\rho}\frac{\partial(\overline{P}+p)}{\partial r}\right)=$$
$$=\left(\frac{\partial(0.5u_r^2)}{\partial t}\right)+\left(\overline{U_r}\frac{\partial(0.5u_r^2)}{\partial r}+u_ru_r\frac{\partial\overline{U_r}}{\partial r}+u_ru_r\frac{\partial u_r}{\partial r}\right)+\left(\frac{\overline{U_\varphi}}{r}\frac{\partial(0.5u_r^2)}{\partial\varphi}+u_ru_\varphi\frac{1}{r}\frac{\partial\overline{U_r}}{\partial\varphi}+u_ru_\varphi\frac{1}{r}\frac{\partial u_r}{\partial\varphi}\right)+ \tag{20}$$
$$+\left(\overline{U_z}\frac{\partial(0.5u_r^2)}{\partial z}+u_ru_z\frac{\partial\overline{U_r}}{\partial z}+u_ru_z\frac{\partial u_r}{\partial z}\right)-\left(u_ru_\varphi\frac{\overline{U_\varphi}}{r}+u_ru_\varphi\frac{\overline{U_\varphi}}{r}+\frac{u_ru_\varphi u_\varphi}{r}\right)+\frac{1}{\rho}u_r\frac{\partial p}{\partial r}$$

$$u_z\left(\frac{\partial(\overline{U_z}+u_z)}{\partial t}+(\overline{U_r}+u_r)\frac{\partial(\overline{U_z}+u_z)}{\partial r}+\frac{(\overline{U_\varphi}+u_\varphi)}{r}\frac{\partial(\overline{U_z}+u_z)}{\partial\varphi}+(\overline{U_z}+u_z)\frac{\partial(\overline{U_z}+u_z)}{\partial z}+\frac{1}{\rho}\frac{\partial(\overline{P}+p)}{\partial z}\right)=$$
$$=\left(\frac{\partial(0.5u_z^2)}{\partial t}\right)+\left(\overline{U_r}\frac{\partial(0.5u_z^2)}{\partial r}+u_zu_r\frac{\partial\overline{U_z}}{\partial r}+u_zu_r\frac{\partial u_z}{\partial r}\right)+\left(\frac{\overline{U_\varphi}}{r}\frac{\partial(0.5u_z^2)}{\partial\varphi}+u_zu_\varphi\frac{1}{r}\frac{\partial\overline{U_z}}{\partial\varphi}+u_zu_\varphi\frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right)+ \tag{21}$$
$$+\left(\overline{U_z}\frac{\partial(0.5u_z^2)}{\partial z}+u_zu_z\frac{1}{r}\frac{\partial U_z}{\partial z}+u_zu_z\frac{1}{r}\frac{\partial u_z}{\partial z}\right)+\frac{1}{\rho}u_z\frac{\partial p}{\partial z}$$

Summing (19)-(21) we obtain part of equation of turbulent kinetic energy transfer.

$$\frac{\partial k}{\partial t}+\overline{U_r}\frac{\partial k}{\partial r}+\frac{\overline{U_\varphi}}{r}\frac{\partial k}{\partial\varphi}+\overline{U_z}\frac{\partial k}{\partial z}+\left[\overline{u_ru_r}\frac{\partial\overline{U_r}}{\partial r}+\overline{u_\varphi u_\varphi}\left(\frac{1}{r}\frac{\partial\overline{U_\varphi}}{\partial\varphi}+\frac{\overline{U_r}}{r}\right)+\overline{u_zu_z}\frac{\partial\overline{U_z}}{\partial z}+\overline{u_\varphi u_r}\left(\frac{\partial\overline{U_\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\overline{U_r}}{\partial\varphi}-\frac{\overline{U_\varphi}}{r}\right)\right.$$
$$\left.+\overline{u_zu_r}\left(\frac{\partial\overline{U_z}}{\partial r}+\frac{\partial\overline{U_r}}{\partial z}\right)+\overline{u_\varphi u_z}\left(\frac{1}{r}\frac{\partial\overline{U_z}}{\partial\varphi}+\frac{\partial\overline{U_\varphi}}{\partial z}\right)\right]+\overline{u_ru_r\frac{\partial u_r}{\partial r}}+\overline{u_\varphi u_\varphi\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}}+\overline{u_zu_z\frac{\partial u_z}{\partial z}}+\overline{u_\varphi u_r\left(\frac{\partial u_\varphi}{\partial r}+\frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right)}+$$
$$+\overline{u_zu_r\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)}+\overline{u_\varphi u_z\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_\varphi}{\partial z}\right)}-\frac{1}{\rho}\left(\frac{1}{r}\overline{u_\varphi\frac{\partial p}{\partial\varphi}}+\overline{u_r\frac{\partial p}{\partial r}}+\overline{u_z\frac{\partial p}{\partial z}}\right)$$

The sum can be written in the form (22). Sense of terms will be described later.

$$\frac{\partial k}{\partial t}+\overline{U_r}\frac{\partial k}{\partial r}+\frac{\overline{U_\varphi}}{r}\frac{\partial k}{\partial\varphi}+\overline{U_z}\frac{\partial k}{\partial z}-P-Diffk_{turb} \tag{22}$$

$$P=-\overline{u_ru_r}\frac{\partial\overline{U_r}}{\partial r}-\overline{u_\varphi u_\varphi}\left(\frac{1}{r}\frac{\partial\overline{U_\varphi}}{\partial\varphi}+\frac{\overline{U_r}}{r}\right)-\overline{u_zu_z}\frac{\partial\overline{U_z}}{\partial z}-\overline{u_\varphi u_r}\left(\frac{\partial\overline{U_\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\overline{U_r}}{\partial\varphi}-\frac{\overline{U_\varphi}}{r}\right)-$$
$$-\overline{u_zu_r}\left(\frac{\partial\overline{U_z}}{\partial r}+\frac{\partial\overline{U_r}}{\partial z}\right)-\overline{u_\varphi u_z}\left(\frac{1}{r}\frac{\partial\overline{U_z}}{\partial\varphi}+\frac{\partial\overline{U_\varphi}}{\partial z}\right)= \tag{23}$$
$$=\nu_t\left[2\left(\frac{\partial\overline{U_r}}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial\overline{U_\varphi}}{\partial\varphi}+\frac{\overline{U_r}}{r}\right)^2+2\left(\frac{\partial\overline{U_z}}{\partial z}\right)^2+\left(\frac{\partial\overline{U_\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\overline{U_r}}{\partial\varphi}-\frac{\overline{U_\varphi}}{r}\right)^2+\left(\frac{\partial\overline{U_z}}{\partial r}+\frac{\partial\overline{U_r}}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial\overline{U_z}}{\partial\varphi}+\frac{\partial\overline{U_\varphi}}{\partial z}\right)^2\right]$$

$$Diffk_{turb} = -\overline{u_r u_r \frac{\partial u_r}{\partial r}} - \overline{u_\varphi u_\varphi \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}} - \overline{u_z u_z \frac{\partial u_z}{\partial z}} - \overline{u_\varphi u_r\left(\frac{\partial u_\varphi}{\partial r} + \frac{1}{r}\frac{\partial u_z}{\partial \varphi}\right)} - \tag{24}$$
$$- \overline{u_z u_r\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)} - \overline{u_\varphi u_z\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z}\right)} - \frac{1}{\rho}\left(\frac{1}{r}\overline{u_\varphi \frac{\partial p}{\partial \varphi}} + \overline{u_r \frac{\partial p}{\partial r}} + \overline{u_z \frac{\partial p}{\partial z}}\right)$$

3. Turbulent kinetic energy transfer, viscous diffusion terms.

Now we transform diffusion terms of (1)-(3).

$$\frac{\mu}{\rho}u_r\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r\left(\overline{U_r}+u_r\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\left(\overline{U_r}+u_r\right)}{\partial \varphi^2} - 2\frac{1}{r^2}\frac{\partial\left(\overline{U_\varphi}+u_\varphi\right)}{\partial \varphi} + \frac{\partial^2\left(\overline{U_r}+u_r\right)}{\partial z^2}\right) =$$
$$= \frac{\mu}{\rho}\left(u_r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r^2}u_r\frac{\partial^2 u_r}{\partial \varphi^2} - 2\frac{1}{r^2}u_r\frac{\partial u_\varphi}{\partial \varphi} + u_r\frac{\partial^2 u_r}{\partial z^2}\right)$$

$$\frac{\mu}{\rho}u_\varphi\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r\left(\overline{U_\varphi}+u_\varphi\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\left(\overline{U_\varphi}+u_\varphi\right)}{\partial \varphi^2} + 2\frac{1}{r^2}\frac{\partial\left(\overline{U_r}+u_r\right)}{\partial \varphi} + \frac{\partial^2\left(\overline{U_\varphi}+u_\varphi\right)}{\partial z^2}\right) =$$
$$= \frac{\mu}{\rho}\left(u_\varphi\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r u_\varphi}{\partial r} + \frac{1}{r^2}u_\varphi\frac{\partial^2 u_\varphi}{\partial \varphi^2} + 2\frac{1}{r^2}u_\varphi\frac{\partial u_r}{\partial \varphi} + u_\varphi\frac{\partial^2 u_\varphi}{\partial z^2}\right)$$

$$\frac{\mu}{\rho}u_z\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\left(\overline{U_z}+u_z\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\left(\overline{U_z}+u_z\right)}{\partial \varphi^2} + \frac{\partial^2\left(\overline{U_z}+u_z\right)}{\partial z^2}\right) =$$
$$= \frac{\mu}{\rho}\left(u_z\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_z}{\partial r} + \frac{1}{r^2}u_z\frac{\partial^2 u_z}{\partial \varphi^2} + u_z\frac{\partial^2 u_z}{\partial z^2}\right)$$

Note, that after opening the brackets derivatives of averaged velocities are found to be multiplied on fluctuation terms. As a result of averaging such prodacts are equal to zero.

Using following formulas:

$$u\frac{\partial^2 u}{\partial \xi^2} = 0.5\frac{\partial u^2}{\partial \xi^2} - \frac{\partial u}{\partial \xi}\frac{\partial u}{\partial \xi};$$

$$u\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} = 0.5\frac{\partial^2 u^2}{\partial r^2} - \frac{\partial u}{\partial r}\frac{\partial u}{\partial r} + \frac{1}{r}0.5\frac{\partial u^2}{\partial r};$$

$$u\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r u}{\partial r} = 0.5\frac{\partial^2 u^2}{\partial r^2} - \frac{\partial u}{\partial r}\frac{\partial u}{\partial r} + \frac{1}{r}0.5\frac{\partial u^2}{\partial r} - \frac{u^2}{r^2}$$

we obtain:

$$u_r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r^2}u_r\frac{\partial^2 u_r}{\partial \varphi^2} - 2\frac{1}{r^2}u_r\frac{\partial u_\varphi}{\partial \varphi} + u_r\frac{\partial^2 u_r}{\partial z^2} =$$
$$= 0.5\frac{\partial^2 u_r^2}{\partial r^2} - \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial r} + \frac{1}{r}0.5\frac{\partial u_r^2}{\partial r} - \frac{u_r^2}{r^2} + \frac{1}{r^2}0.5\frac{\partial u_r^2}{\partial \varphi^2} - \frac{1}{r^2}\frac{\partial u_r}{\partial \varphi}\frac{\partial u_r}{\partial \varphi} - 2\frac{1}{r^2}u_r\frac{\partial u_\varphi}{\partial \varphi} + 0.5\frac{\partial^2 u_r^2}{\partial z^2} - \frac{\partial u_r}{\partial z}\frac{\partial u_r}{\partial z}$$

$$u_\varphi \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r u_\varphi}{\partial r} + \frac{1}{r^2} u_\varphi \frac{\partial^2 u_\varphi}{\partial \varphi^2} + 2\frac{1}{r^2} u_\varphi \frac{\partial u_r}{\partial \varphi} + u_\varphi \frac{\partial^2 u_\varphi}{\partial z^2} =$$
$$= 0.5\frac{\partial^2 u_\varphi^2}{\partial r^2} - \frac{\partial u_\varphi}{\partial r}\frac{\partial u_\varphi}{\partial r} + \frac{1}{r}0.5\frac{\partial u_\varphi^2}{\partial r} - \frac{u_\varphi^2}{r^2} + \frac{1}{r^2}0.5\frac{\partial u_\varphi^2}{\partial \varphi^2} - \frac{1}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\frac{\partial u_\varphi}{\partial \varphi} + 2\frac{1}{r^2} u_\varphi \frac{\partial u_r}{\partial \varphi} + 0.5\frac{\partial^2 u_\varphi^2}{\partial z^2} - \frac{\partial u_\varphi}{\partial z}\frac{\partial u_\varphi}{\partial z}$$

$$u_z \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u_z}{\partial r} + \frac{1}{r^2} u_z \frac{\partial^2 u_z}{\partial \varphi^2} + u_z \frac{\partial^2 u_z}{\partial z^2} =$$
$$= 0.5\frac{\partial^2 u_z^2}{\partial r^2} - \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial r} + \frac{1}{r}0.5\frac{\partial u_z^2}{\partial r} + \frac{1}{r^2}0.5\frac{\partial u_z^2}{\partial \varphi^2} - \frac{1}{r^2}\frac{\partial u_z}{\partial \varphi}\frac{\partial u_z}{\partial \varphi} + 0.5\frac{\partial^2 u_z^2}{\partial z^2} - \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial z}.$$

Summing and regrouping terms we obtain:

$$0.5\frac{\partial^2 u_r^2}{\partial r^2} + \frac{1}{r}0.5\frac{\partial u_r^2}{\partial r} + \frac{1}{r^2}0.5\frac{\partial u_r^2}{\partial \varphi^2} + 0.5\frac{\partial^2 u_r^2}{\partial z^2} - \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}\frac{\partial u_r}{\partial \varphi}\frac{\partial u_r}{\partial \varphi} - \frac{\partial u_r}{\partial z}\frac{\partial u_r}{\partial z} - 2\frac{1}{r^2} u_r \frac{\partial u_\varphi}{\partial \varphi} - \frac{u_r^2}{r^2} +$$
$$+ 0.5\frac{\partial^2 u_\varphi^2}{\partial r^2} + \frac{1}{r}0.5\frac{\partial u_\varphi^2}{\partial r} + \frac{1}{r^2}0.5\frac{\partial u_\varphi^2}{\partial \varphi^2} + 0.5\frac{\partial^2 u_\varphi^2}{\partial z^2} - \frac{\partial u_\varphi}{\partial r}\frac{\partial u_\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\frac{\partial u_\varphi}{\partial \varphi} - \frac{\partial u_\varphi}{\partial z}\frac{\partial u_\varphi}{\partial z} + 2\frac{1}{r^2} u_\varphi \frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r^2} +$$
$$+ 0.5\frac{\partial^2 u_z^2}{\partial r^2} + \frac{1}{r}0.5\frac{\partial u_z^2}{\partial r} + \frac{1}{r^2}0.5\frac{\partial u_z^2}{\partial \varphi^2} + 0.5\frac{\partial^2 u_z^2}{\partial z^2} - \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial r} - \frac{1}{r^2}\frac{\partial u_z}{\partial \varphi}\frac{\partial u_z}{\partial \varphi} - \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial z} =$$
$$= \frac{\partial^2 k}{\partial r^2} + \frac{1}{r}\frac{\partial k}{\partial r} + \frac{1}{r^2}\frac{\partial k}{\partial \varphi^2} + \frac{\partial^2 k}{\partial z^2} - \varepsilon = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial k}{\partial r} + \frac{1}{r^2}\frac{\partial k}{\partial \varphi^2} + \frac{\partial^2 k}{\partial z^2} - \varepsilon$$

Here:

$$\varepsilon = -\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}\frac{\partial u_r}{\partial \varphi}\frac{\partial u_r}{\partial \varphi} - \frac{\partial u_r}{\partial z}\frac{\partial u_r}{\partial z} - 2\frac{1}{r^2} u_r \frac{\partial u_\varphi}{\partial \varphi} - \frac{u_r^2}{r^2} - \frac{\partial u_\varphi}{\partial r}\frac{\partial u_\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\frac{\partial u_\varphi}{\partial \varphi} - \frac{\partial u_\varphi}{\partial z}\frac{\partial u_\varphi}{\partial z} +$$
$$+ 2\frac{1}{r^2} u_\varphi \frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r^2} - \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial r} - \frac{1}{r^2}\frac{\partial u_z}{\partial \varphi}\frac{\partial u_z}{\partial \varphi} - \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial z}$$

Collecting all terms of all equation we obtain:

$$\frac{\partial k}{\partial t} + \overline{U_r}\frac{\partial k}{\partial r} + \frac{\overline{U_\varphi}}{r}\frac{\partial k}{\partial \varphi} + \overline{U_z}\frac{\partial k}{\partial z} = \nu\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial k}{\partial r} + \frac{1}{r^2}\frac{\partial k}{\partial \varphi^2} + \frac{\partial^2 k}{\partial z^2}\right) + Diffk_{turb} + P - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{U_r}\frac{\partial \varepsilon}{\partial r} + \frac{\overline{U_\varphi}}{r}\frac{\partial \varepsilon}{\partial \varphi} + \overline{U_z}\frac{\partial \varepsilon}{\partial z} = \nu\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \varepsilon}{\partial r} + \frac{1}{r^2}\frac{\partial \varepsilon}{\partial \varphi^2} + \frac{\partial^2 \varepsilon}{\partial z^2}\right) + Diff\varepsilon_{turb} + \frac{\varepsilon}{k}\left(C_1 P - C_2 \varepsilon\right)$$

$$\nu_t = C_\nu \frac{k^2}{\varepsilon}$$

Here P – turbulent energy production an is given by (23), задается соотношением (), ε - rate of k dissipation, $Diffk_{turb}$ и $Diff\varepsilon_{turb}$ - turbulent diffusion of k and ε, depends on concrete model, ν_t – turbulent viscosity, C_1, C_2, C_ν - model constants.

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Анотація

Б. Головня

ПОВНА k-ε МОДЕЛЬ ТУРБУЛЕНТНОСТІ В ЦИЛІНДРИЧНИХ КООРДИНАТАХ

Ламінарні рівняння Нав'є-Стокса в циліндричних координатах описані в більшості книг з теоретичної гідродинаміки [1] - [3]. Рівняння Рейнольдса також легко знайти в літературі. Але повні рівняння εk моделі турбулентності в циліндричних координатах зустрічаються дуже рідко. Як правило, ці рівняння наводяться в роботах лише в спрощеному вигляді. У даній статті отримана не спрощена система рівнянь k-ε моделі турбулентності в циліндричних координатах

Ключові слова: *k-ε модель турбулентності, рівняння Рейнольдсаб циліндричні координати*

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